Spectral gap for some spherically symmetric probability measures

Aldéric Joulin
Institut de Mathématiques de Toulouse

Joint with Michel Bonnefont and Yutao Ma

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Outline

1. Gaussian case
2. Log-concave case

Consider standard Gaussian probability measure γ on Euclidean state space $(\mathbb{R}^n, \|\cdot\|)$:

$$\gamma(dx) = (2\pi)^{-n/2}\, e^{-\|x\|^2/2}\, dx.$$

Invariant measure of Ornstein-Uhlenbeck process solution to SDE

$$\begin{cases} dX_t^x &= \sqrt{2}\, dB_t - X_t^x\, dt \\ X_0^x &= x \end{cases}$$

where $(B_t)_{t\geq 0}$ Brownian motion in $\mathbb{R}^n$.

Associated semigroup gives the distribution at time $t > 0$ of the process: for $f \in \mathcal{C}_b(\mathbb{R}^n)$,

$$P_t f(x) := \mathbb{E}[f(X_t^x)] \;=\; \int_{\mathbb{R}^n} f(xe^{-t} + \sqrt{1-e^{-2t}}\,y)\,\gamma(dy).$$

Markov semigroup:

$$P_t \circ P_s = P_s \circ P_t = P_{t+s} \quad \text{and} \quad P_0 = \mathrm{Id}.$$

Invariance with respect to γ:

$$\int_{\mathbb{R}^n} P_t f \, d\gamma = \int_{\mathbb{R}^n} f \, d\gamma =: \gamma(f).$$

Long-time convergence to equilibrium: for any $x \in \mathbb{R}^n$,

$$P_t f(x) \underset{t \to +\infty}{\longrightarrow} \gamma(f), \quad f \in \mathcal{C}_b(\mathbb{R}^n).$$

Questions:

- speed of convergence ? (concerns time variable t)
- In which space ? (concerns space variable x)

Associated generator (derivative at time $t = 0$ of the semigroup):

$$\mathcal{L}f(x) = \Delta f(x) - x \cdot \nabla f(x).$$

As a diffusion operator, $\mathcal{L}$ is non-positive and self-adjoint in $L^2(\gamma)$:

$$\int_{\mathbb{R}^n} f \, \mathcal{L}g \, d\gamma = \int_{\mathbb{R}^n} \mathcal{L}f \, g \, d\gamma = - \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\gamma.$$

Poincaré inequality with constant $\lambda > 0$: for all smooth $f : \mathbb{R}^n \to \mathbb{R}$,

$$\lambda \, \mathrm{Var}_\gamma(f) \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 \, d\gamma \qquad \text{PI}(\lambda)$$

where $\mathrm{Var}_\gamma(f) := \gamma(f^2) - \gamma(f)^2$.

PI(λ) rewrites as

$$\lambda \leq \int_{\mathbb{R}^n} f(-\mathcal{L}f)\, d\gamma,$$

for all smooth f such that $\gamma(f) = 0$ and $\gamma(f^2) = 1$.

Optimal constant λ given by variational formula

$$\inf \left\{ \int_{\mathbb{R}^n} f(-\mathcal{L}f)\, d\gamma : f \text{ smooth},\ \gamma(f) = 0,\ \gamma(f^2) = 1 \right\}.$$

We have:

$$\mathrm{PI}(\lambda) \quad \Longleftrightarrow \quad \text{Spectrum}(-\mathcal{L}) \subset \{0\} \cup [\lambda, \infty).$$

Optimal constant denoted $\lambda_1(-\mathcal{L}, \gamma)$: spectral gap of $-\mathcal{L}$.

Link with convergence to equilibrium in $L^2(\gamma)$:

$$\mathrm{PI}(\lambda) \iff \|P_t f - \gamma(f)\|_{L^2(\gamma)} \overset{(\star)}{\leq} e^{-\lambda t} \, \|f - \gamma(f)\|_{L^2(\gamma)}, \, \forall f \text{ smooth.}$$

Proof:

Inequality $(\star)$ rewrites as

$$\mathrm{Var}_\gamma(P_t f) \leq e^{-2\lambda t} \, \mathrm{Var}_\gamma(f), \quad \forall f \text{ smooth.}$$

Let f smooth such that $\gamma(f) = 0$ and denote

$$\phi(t) := \int_{\mathbb{R}^n} (P_t f)^2 \, d\gamma.$$

$\Longleftarrow$: inequality $(\star)$ means

$$\phi(t) \leq e^{-2\lambda t}\, \phi(0),$$

which implies

$$\phi'(0) \leq -2\lambda\, \phi(0),$$

i.e., PI(λ).

$\Longrightarrow$: apply PI(λ) with $P_t f$ (instead of f) to get

$$\phi'(t) \leq -2\lambda\, \phi(t),$$

and Gronwall's lemma entails

$$\phi(t) \leq e^{-2\lambda t}\, \phi(0),$$

i.e. inequality $(\star)$.

Two important facts :

• Standard Gaussian measure is a product measure.

• Poincaré inequality tensorizes, i.e, Poincaré inequality for a one-dimensional measure entails Poincaré inequality for the product measure with the same constant.

$\Longrightarrow$ The reason why in the Gaussian case, if $\lambda_1(-\mathcal{L},\gamma) > 0$, then it should not depend on dimension n.

Theorem

In the Gaussian case, we have

$$\textit{Spectrum } (-\mathcal{L}, \gamma) = \mathbb{N},$$

and the eigenfunctions are the Hermite polynomials. In particular $\lambda_1(-\mathcal{L}, \gamma) = 1$ *and the associated eigenfunction is linear.*

Generalization of Gaussian measure: exponential power distribution of parameter $\alpha > 0$,

$$\mu(dx) = \frac{1}{Z} e^{-\|x\|^\alpha/\alpha} dx, \quad x \in \mathbb{R}^n.$$

- $\alpha = 1$: exponential measure.
- $\alpha = 2$: Gaussian case.
- $\alpha \to \infty$: uniform distribution on the Euclidean unit ball.

$\Longrightarrow$ Product measure iff $\alpha = 2$.

Associated generator:

$$\mathcal{L}f(x) = \begin{cases} \Delta f(x) - \|x\|^{\alpha-2} x \cdot \nabla f(x) & \text{on} \quad \mathbb{R}^n \quad \text{if} \quad 0 < \alpha < \infty; \\ \Delta f(x) & \text{on} \quad \mathcal{B}_n \quad \text{if} \quad \alpha = \infty, \end{cases}$$

the latter endowed with Neumann's conditions at the boundary $\mathbb{S}^{n-1}$.

A general result:

Theorem

$$\lambda_1(-\mathcal{L}, \mu) > 0 \quad \Longleftrightarrow \quad \alpha \geq 1.$$

In particular, what is the dependence of $\lambda_1(-\mathcal{L}, \mu)$ w.r.t. dimension n ?

For $n = 1$, the spectral gap is known only for:

- $\alpha = 1$: $\quad \lambda_1(-\mathcal{L}, \mu) = 1/4.$
- $\alpha = 2$: $\quad \lambda_1(-\mathcal{L}, \mu) = 1.$
- $\alpha \to \infty$: $\quad \lambda_1(-\mathcal{L}, \mu) = \pi^2/4.$

A measure μ is called:

• log-concave if

$$\mu(dx) = \frac{1}{Z}\, e^{-V(x)}\, dx,$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is convex.

• spherically symmetric log-concave (including exponential power distributions) if

$$\mu(dx) = \frac{1}{Z}\, e^{-V(\|x\|)}\, dx,$$

where $V : \mathbb{R}_+ \to \mathbb{R}$ convex and non-decreasing.

Associated generator:

$$\mathcal{L}f = \Delta f - \nabla V \cdot \nabla f.$$

Famous Bakry-Émery criterion: if there exists $\lambda > 0$ such that

$$\text{Hess } V(x) \geq \lambda \, \text{Id}, \quad x \in \mathbb{R}^n,$$

then

$$\lambda_1(-\mathcal{L}, \mu) \geq \lambda.$$

$\Longrightarrow$ If V is only convex (i.e. $\lambda = 0$ above) then BE criterion fails: case of exponential power distribution of parameter $\alpha \neq 2$.

An important conjecture:

Conjecture (Kannan, Lovász and Simonovits, 1995)

There exists a universal constant $C > 0$, independent from dimension n, such that any isotropic log-concave measure (i.e. $\int_{\mathbb{R}^n} |x \cdot \theta|^2 \, \mu(dx) = \|\theta\|^2$ for all $\theta \in \mathbb{R}^n$) satisfies

$$C \leq \lambda_1(-\mathcal{L}, \mu) \leq 1.$$

Still open, a large field of investigation.

In 2003, Bobkov focuses on the spherically symmetric case.

Theorem (Bobkov, 2003)

If μ is spherically symmetric log-concave in $\mathbb{R}^n$, $n \geq 1$, then

$$\frac{n}{13 \int_{\mathbb{R}^n} \|x\|^2 \mu(dx)} \leq \lambda_1(-\mathcal{L}, \mu) \leq \frac{n}{\int_{\mathbb{R}^n} \|x\|^2 \mu(dx)}.$$

In particular, spherically symmetric log-concave measures satisfies the KLS conjecture, the isotropic condition meaning in our context

$$\int_{\mathbb{R}^n} \|x\|^2 \mu(dx) = n.$$

Our recent improvement of Bobkov's result:

Theorem (Bonnefont-Joulin-Ma, 2014)

If μ is spherically symmetric log-concave in $\mathbb{R}^n$, $n \geq 2$, then

$$\frac{n-1}{\int_{\mathbb{R}^n} \|x\|^2 \, \mu(dx)} \leq \lambda_1(-\mathcal{L}, \mu) \leq \frac{n}{\int_{\mathbb{R}^n} \|x\|^2 \, \mu(dx)}.$$

In particular

$$\lambda_1(-\mathcal{L}, \mu) \underset{n \to \infty}{\sim} \frac{n}{\int_{\mathbb{R}^n} \|x\|^2 \, \mu(dx)}.$$

Idea of the proof, based on Bobkov's approach:

Upper bound: trivial by taking in variational formula

$$\lambda_1(-\mathcal{L}, \mu) = \inf \left\{ \frac{\int_{\mathbb{R}^n} f\,(-\mathcal{L}f)\,d\mu}{\mathrm{Var}_\mu(f)} : f \text{ smooth }, f \neq \text{const} \right\},$$

the linear function $f(x) = x \cdot \mathbb{1}$.

Lower bound:

• Comparison with the spectral gap of the underlying one-dimensional radial part (a slight improvement).

• Spectral estimate for the radial part (an important improvement).

Back to exponential power distribution of parameter $\alpha \geq 1$:

$$\mu(dx) = \frac{1}{Z}\, e^{-\|x\|^\alpha/\alpha}\, dx, \quad x \in \mathbb{R}^n.$$

Recall associated generator:

$$\mathcal{L}f(x) = \begin{cases} \Delta f(x) - \|x\|^{\alpha-2}\, x \cdot \nabla f(x) & \text{on} \quad \mathbb{R}^n \quad \text{if} \quad 0 < \alpha < \infty; \\ \Delta f(x) & \text{on} \quad \mathcal{B}_n \quad \text{if} \quad \alpha = \infty, \end{cases}$$

the latter endowed with Neumann's boundary conditions.

Corollary

For fixed $\alpha \in [1, \infty]$,

$$\lambda_1(-\mathcal{L}, \mu) \underset{n \to \infty}{\sim} n^{1-2/\alpha}.$$