Journées de Probabilités

Structures de régularité et renormalisation d'EDPS de FitzHugh–Nagumo

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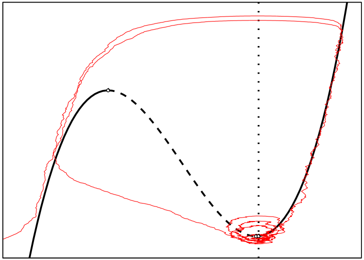
avec Christian Kuehn (Vienne)

FitzHugh–Nagumo SDE

$$
\begin{aligned}
\mathrm{d}u_t &= \frac{1}{\varepsilon}[u_t - u_t^3 + v_t]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \\
\mathrm{d}v_t &= [a - u_t - bv_t]\,\mathrm{d}t
\end{aligned}
$$

- u_t: membrane potential of neuron
- v_t: gating variable (proportion of open ion channels)

$\varepsilon = 0.1$
$b = 0$
$\delta = a - \frac{1}{\sqrt{3}} = 0.02$
$\sigma = 0.03$



FitzHugh–Nagumo SPDE

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$
$$\partial_t v = a_1 u + a_2 v$$

- $u = u(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}^n$, $(t,x) \in D = \mathbb{R}_+ \times \mathbb{T}^d$, $d = 2, 3$
- $\xi(t,x)$ Gaussian space-time white noise: $\mathbb{E}\big[\xi(t,x)\xi(s,y)\big] = \delta(t-s)\delta(x-y)$

  ξ: distribution defined by $\langle \xi, \varphi \rangle = W_\varphi$, $\{W_h\}_{h \in L^2(D)}$, $\mathbb{E}[W_h W_{h'}] = \langle h, h' \rangle$

(Link to simulation)

Main result

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$
where $\varrho_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+2}}\varrho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ with ρ compactly supported, integral 1

Theorem [NB & C. Kuehn, preprint 2015, arXiv/1504.02953]

There exists a choice of renormalisation constant $C(\varepsilon)$, $\lim_{\varepsilon\to 0} C(\varepsilon) = \infty$, such that

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + [1 + C(\varepsilon)]u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon + \xi^\varepsilon$$
$$\partial_t v^\varepsilon = a_1 u^\varepsilon + a_2 v^\varepsilon$$

admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$.

- Local solution means up to a random possible explosion time
- Initial conditions should be in appropriate Hölder spaces
- $C(\varepsilon) \asymp \log(\varepsilon^{-1})$ for $d = 2$ and $C(\varepsilon) \asymp \varepsilon^{-1}$ for $d = 3$
- Similar results for general cubic nonlinearity and $v \in \mathbb{R}^n$

Mild solutions of SPDE

$$\partial_t u = \Delta u + F(u) + \xi$$

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Construction of mild solution via Duhamel formula:

- $\partial_t u = \Delta u \quad \Rightarrow \quad u(t,x) = \int G(t,x-y)u_0(y)\,\mathrm{d}y =: (\mathrm{e}^{\Delta t}\,u_0)(x)$

  where $G(t,x)$: heat kernel (compatible with bc)

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- $\partial_t u = \Delta u + f \quad \Rightarrow \quad u(t,x) = (\mathrm{e}^{\Delta t}\,u_0)(x) + \displaystyle\int_0^t \mathrm{e}^{\Delta(t-s)}\,f(s,\cdot)(x)\,\mathrm{d}s$

  Notation: $u = Gu_0 + G * f$

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- $\partial_t u = \Delta u + \xi \quad \Rightarrow \quad u = Gu_0 + G * \xi \qquad$ (stochastic convolution)

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- $\partial_t u = \Delta u + \xi + F(u) \quad \Rightarrow \quad u = Gu_0 + G * [\xi + F(u)]$

  Aim: use Banach's fixed-point theorem — but which function space?

Hölder spaces

Definition of $\mathcal{C}^\alpha$ for $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ a compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha-1}$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$

  where $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$ for all test functions $\eta \in \mathcal{C}^{-\lfloor \alpha \rfloor}$

Property: $f \in \mathcal{C}^\alpha$, $0 < \alpha < 1 \quad \Rightarrow \quad f' \in \mathcal{C}^{\alpha-1}$ where $\langle f', \eta \rangle = -\langle f, \eta' \rangle$

Remark: $f \in \mathcal{C}^{1+\alpha} \not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^{1+\alpha}$. See e.g $f(x) = x + |x|^{3/2}$

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Case of the heat kernel: $(\partial_t - \Delta)u = f \quad \Rightarrow \quad u = G * f$

Parabolic scaling $\mathcal{C}_\mathfrak{s}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

$$\frac{1}{\delta}\eta\left(\frac{x-y}{\delta}\right) \longrightarrow \frac{1}{\delta^{d+2}}\eta\left(\frac{t-s}{\delta^2}, \frac{x-y}{\delta}\right)$$

Schauder estimates and fixed-point equation

Schauder estimate

$f \in \mathcal{C}_{\mathfrak{s}}^{\alpha} \quad \Rightarrow \quad G * f \in \mathcal{C}_{\mathfrak{s}}^{\alpha+2}$

Fact: in dimension d, space-time white noise $\xi \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$ a.s. $\forall \alpha < -\frac{d+2}{2}$

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Fixed-point equation: $u = Gu_0 + G * [\xi + F(u)]$

- $d = 1$: $\xi \in \mathcal{C}^{-3/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ defined
- $d = 3$: $\xi \in \mathcal{C}^{-5/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{-1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined
- $d = 2$: $\xi \in \mathcal{C}^{-2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{0^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined

  Boundary case, can be treated with Besov spaces
  [Da Prato and Debussche 2003]

Why not use mollified noise? Limit $\varepsilon \to 0$ does not exist

Regularity structures

Basic idea of Martin Hairer [Inventiones Mathematicae, 2014]:

Lift mollified fixed-point equation

$$u = Gu_0 + G * [\xi^\varepsilon + F(u)]$$

to a larger space called a Regularity structure

$$\begin{tikzcd}
(u_0, Z^\varepsilon) \arrow[r, mapsto, "\mathcal{S}"] & U \arrow[d, mapsto, "\mathcal{R}"] \\
(u_0, \xi^\varepsilon) \arrow[u, mapsto, "\Psi"] \arrow[r, mapsto, "\bar{\mathcal{S}}"'] & u^\varepsilon
\end{tikzcd}$$

- $u^\varepsilon = \bar{\mathcal{S}}(u_0, \xi^\varepsilon)$: classical solution of mollified equation
- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)

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- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)
- Renormalisation: modification of the lift Ψ

Basic idea: Generalised Taylor series

$f : I \to \mathbb{R}$, $0 < \alpha < 1$

$f \in \mathcal{C}^{2+\alpha} \quad \Leftrightarrow \quad f \in \mathcal{C}^2$ and $f'' \in \mathcal{C}^\alpha$

Associate with f the triple (f, f', f'')

When does a triple (f_0, f_1, f_2) represent a function $f \in \mathcal{C}^{2+\alpha}$?

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When there is a constant C such that for all $x, y \in I$

$$\begin{aligned}
\left|f_0(y) - f_0(x) - (y-x)f_1(x) - \tfrac{1}{2}(y-x)^2 f_2(x)\right| &\leqslant C|x-y|^{2+\alpha} \\
\left|f_1(y) - f_1(x) - (y-x)f_2(x)\right| &\leqslant C|x-y|^{1+\alpha} \\
\left|f_2(y) - f_2(x))\right| &\leqslant C|x-y|^{\alpha}
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|f_1(y) - f_1(x) - (y-x)f_2(x)| &\leqslant C|x-y|^{1+\alpha} \\
|f_2(y) - f_2(x))| &\leqslant C|x-y|^{\alpha}
\end{aligned}
$$

Notation: $f = f_0\mathbf{1} + f_1 X + f_2 X^2$

Regularity structure: Generalised Taylor basis whose basis elements can also be singular distributions

Definition of a regularity structure

Definition [M. Hairer, Inventiones Math 2014]

A Regularity structure is a triple $(A, T, \mathcal{G})$ where

1. Index set: $A \subset \mathbb{R}$, bdd below, locally finite, $0 \in A$
2. Model space: $T = \bigoplus_{\alpha \in A} T_\alpha$, each T_α Banach space, $T_0 = \mathrm{span}(\mathbf{1}) \simeq \mathbb{R}$
3. Structure group: $\mathcal{G}$ group of linear maps $\Gamma : T \to T$ such that

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta \qquad \forall \tau \in T_\alpha$$

   and $\Gamma\mathbf{1} = \mathbf{1}\ \forall \Gamma \in \mathcal{G}$.

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and $\Gamma\mathbf{1} = \mathbf{1}$ $\forall \Gamma \in \mathcal{G}$.

Polynomial regularity structure on $\mathbb{R}$:

- $A = \mathbb{N}_0$
- $T_k \simeq \mathbb{R}$, $T_k = \mathrm{span}(X^k)$
- $\Gamma_h(X^k) = (X-h)^k$ $\forall h \in \mathbb{R}$

Polynomial reg. structure on $\mathbb{R}^d$: $X^k = X_1^{k_1} \dots X_d^{k_d} \in T_{|k|}$, $|k| = \sum_{i=1}^d k_i$

Regularity structure for $\partial_t u = \Delta u - u^3 + \xi$

New symbols: Ξ, representing ξ, Hölder exponent $|\Xi|_{\mathfrak{s}} = \alpha_0 = -\frac{d+2}{2} - \kappa$

$\mathcal{I}(\tau)$, representing $G * f$, Hölder exponent $|\mathcal{I}(\tau)|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + 2$

$\tau\sigma$, Hölder exponent $|\tau\sigma|_{\mathfrak{s}} = |\tau|_{\mathfrak{s}} + |\sigma|_{\mathfrak{s}}$

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τ	Symbol	$\lvert\tau\rvert_{\mathfrak{s}}$	$d = 3$	$d = 2$
Ξ	Ξ	α_0	$-\frac{5}{2} - \kappa$	$-2 - \kappa$
$\mathcal{I}(\Xi)^3$		$3\alpha_0 + 6$	$-\frac{3}{2} - 3\kappa$	$0 - 3\kappa$
$\mathcal{I}(\Xi)^2$		$2\alpha_0 + 4$	$-1 - 2\kappa$	$0 - 2\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2$		$5\alpha_0 + 12$	$-\frac{1}{2} - 5\kappa$	$2 - 5\kappa$
$\mathcal{I}(\Xi)$		$\alpha_0 + 2$	$-\frac{1}{2} - \kappa$	$0 - \kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\Xi)^2 X_i$	 X_i	$2\alpha_0 + 5$	$0 - 2\kappa$	$1 - 2\kappa$
$\mathbf{1}$	$\mathbf{1}$	0	0	0
$\mathcal{I}(\mathcal{I}(\Xi)^3)$		$3\alpha_0 + 8$	$\frac{1}{2} - 3\kappa$	$2 - 3\kappa$
...	...	...	...	...

Fixed-point equation for $\partial_t u = \Delta u - u^3 + \xi$

$$u = G * [\xi^\varepsilon - u^3] + G u_0 \qquad \Rightarrow \qquad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \dots$$

$$\begin{aligned} U_0 &= 0 \\ U_1 &= \; + \varphi \mathbf{1} \\ U_2 &= \; + \varphi \mathbf{1} - \; + 3\varphi \; + \dots \end{aligned}$$

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$$\begin{aligned} U_0 &= 0 \\ U_1 &= \mathcal{I}(\Xi) + \varphi\mathbf{1} \\ U_2 &= \mathcal{I}(\Xi) + \varphi\mathbf{1} - \mathcal{I}(\mathcal{I}(\Xi)^3) + 3\varphi\,\mathcal{I}(\mathcal{I}(\Xi)^2) + \dots \end{aligned}$$

To prove convergence, we need

- A model (Π, Γ): $\forall z \in \mathbb{R}^{d+1}$, $\Pi_z \tau$ is distribution describing τ near z
  $\Gamma_{z\bar{z}} \in \mathcal{G}$ describes translations: $\Pi_{\bar{z}} = \Pi_z \Gamma_{z\bar{z}}$

- Spaces
  $$\mathcal{D}^\gamma = \Big\{ f : \mathbb{R}^{d+1} \to \bigoplus_{\beta<\gamma} T_\beta : \|f(z) - \Gamma_{z\bar{z}} f(\bar{z})\|_\beta \lesssim \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta} \Big\}$$
  equipped with a seminorm

- The Reconstruction theorem: provides a unique map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}^\alpha_{\mathfrak{s}}$
  such that $|\langle \mathcal{R}f - \Pi_z f(z), \eta^\delta_{\mathfrak{s},z} \rangle| \lesssim \delta^\gamma$ (constructed using wavelets)

Why do we need to renormalise?

Let $G_\varepsilon = G * \varrho_\varepsilon$ where ϱ_ε is the mollifier

$$(\Pi_{\bar{z}} \overset{\bullet}{|})(z) = (G * \xi^\varepsilon)(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1)\xi(z_1)\,\mathrm{d}z_1$$

belongs to first Wiener chaos, limit $\varepsilon \to 0$ well-defined

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$$(\Pi_{\bar z}\,)(z) = (G * \xi^\varepsilon)(z)^2 = \iint G_\varepsilon(z - z_1)G_\varepsilon(z - z_2)\xi(z_1)\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

diverges as $\varepsilon \to 0$

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diverges as $\varepsilon \to 0$

Wick product: $\xi(z_1) \diamond \xi(z_2) = \xi(z_1)\xi(z_2) - \delta(z_1 - z_2)$

$$(\Pi_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) = \underbrace{\iint G_\varepsilon(z - z_1)G_\varepsilon(z - z_2)\xi(z_1) \diamond \xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2}_{\text{in 2nd Wiener chaos, bdd}} + \underbrace{\int G_\varepsilon(z - z_1)^2\,\mathrm{d}z_1}_{C_1(\varepsilon)\to\infty}$$

Renormalised model: $(\widehat{\Pi}_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) = (\Pi_{\bar z} \overset{\bullet\,\bullet}{\vee})(z) - C_1(\varepsilon)$

The case of the FitzHugh–Nagumo equations

Fixed-point equation

$$u(t,x) = G * [\xi^\varepsilon + u - u^3 + v](t,x) + Gu_0(t,x)$$
$$v(t,x) = \int_0^t u(s,x)\,\mathrm{e}^{(t-s)a_2}\,a_1\,\mathrm{d}s + \mathrm{e}^{ta_2}\,v_0$$

Lifted version

$$U = \mathcal{I}[\Xi + U - U^3 + V] + Gu_0$$
$$V = \mathcal{E}U + Qv_0$$

where $\mathcal{E}$ is an integration map which is not regularising in space
New symbols $\mathcal{E}(\mathcal{I}(\Xi)) =$, etc...

We expect U, and thus also V to be α-Hölder for $\alpha < -\frac{1}{2}$
Thus $\mathcal{I}(U - U^3 + V)$ should be well-defined

The standard theory has to be extended, because $\mathcal{E}$ does not correspond to a smooth kernel

Concluding remarks

- Models with $\partial_t u$ of order $u^4 + v^4$ and $\partial_t v$ of order $u^2 + v$ should be renormalisable
  Current approach does not work when singular part (t, x)-dependent
- Global existence: recent progress by Hendrik Weber on 2D Allen–Cahn
- More quantitative results?

References

- Martin Hairer, *A theory of regularity structures*, Invent. Math. **198** (2), pp 269–504 (2014)
- Martin Hairer, *Introduction to Regularity Structures*, lecture notes (2013)
- N. B., Christian Kuehn, *Regularity structures and renormalisation of FitzHugh-Nagumo SPDEs in three space dimensions*, preprint `arXiv/1504.02953`